Test 2
(Vectors and Mathematical Reasoning)

This assessment contributes 5% towards the final year mark.
40 minutes are allocated for this task.
The use of a CAS calculator is assumed for this task.

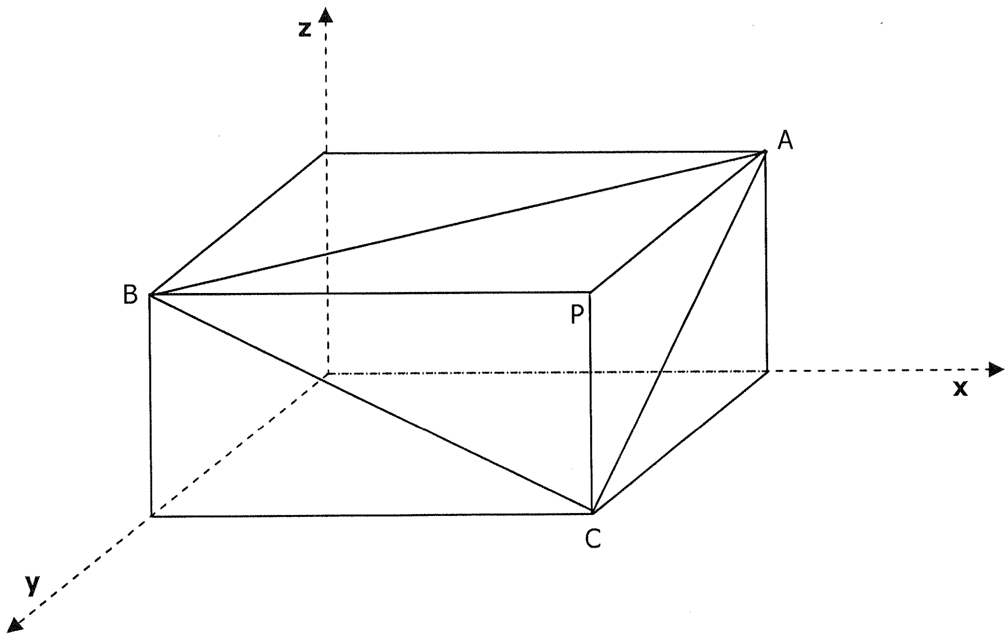
Name : SOLUTIONS / Marking Key

Score : (out of 50)

Note : There are spare pages at the back of this booklet, if you require additional working space.

Do NOT turn over this page until you are instructed to do so.

1. The diagram shows a rectangular prism with dimensions 10 cm x 8 cm x 5 cm. Point P has position vector 10**i** + 8**j** + 5**k**.



a. Determine the position vectors for points A, B and C.

$\underset{\sim}{a} = \begin{pmatrix}10\\0\\5\end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix}0\\8\\5\end{pmatrix}, \quad \underset{\sim}{c} = \begin{pmatrix}10\\8\\0\end{pmatrix}$

✓✓ 3 correct
✓ 2 correct

[2]

b. Determine the vector equation of the line containing points A and C.

$\underset{\sim}{d} = \overrightarrow{AC} = \begin{pmatrix}0\\8\\-5\end{pmatrix}$

$\therefore \underset{\sim}{r} = \begin{pmatrix}10\\0\\5\end{pmatrix} + \lambda \begin{pmatrix}0\\8\\-5\end{pmatrix} = \begin{pmatrix}10\\8\lambda\\5-5\lambda\end{pmatrix}$ ✓ ✓

[3]

✓ Form of equation

Consider the plane determined by the points A, B and C. Hereafter in this question, this is the plane referred to as "the plane". Shade the plane in the diagram above.

c. Show that the normal vector to the plane can be given by **n** = 4**i** + 5**j** + 8**k**.

$\underset{\sim}{n}$ is perpendicular to any vector in the plane.

$\therefore \underset{\sim}{n} \perp \overrightarrow{AC}$ ✓

$\therefore \begin{pmatrix}4\\5\\8\end{pmatrix} \cdot \begin{pmatrix}0\\8\\-5\end{pmatrix}$ ✓ $= 0 + 40 - 40$

$= 0$ ✓

$\therefore \begin{pmatrix}4\\5\\8\end{pmatrix}$ is a normal to the plane

Σ 8m

[3]

1. Given that point R is any point in the plane having position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

d. Show that $4x + 5y + 8z = 80$.

Eq$^{\underline{n}}$ plane : $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{n} \cdot \underset{\sim}{a}$ ✓

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix} ✓ = \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 0 \\ 5 \end{pmatrix}$$

$$\therefore \quad 4x + 5y + 8z = 40 + 40 ✓ = 80$$

[3]

e. Give the expression for the vector $\overrightarrow{PR}$

$$\overrightarrow{PR} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 10 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} x-10 \\ y-8 \\ z-5 \end{pmatrix} ✓$$

[1]

f. Find the value of k such that $\overrightarrow{PR} = k\,\mathbf{n}$

i.e. $\overrightarrow{PR}$ is parallel to the normal vector $\mathbf{n}$

$$\begin{pmatrix} x-10 \\ y-8 \\ z-5 \end{pmatrix} = k\begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 4k \\ 5k \\ 8k \end{pmatrix} \qquad \therefore \quad x = 10 + 4k,\; y = 8 + 5k,\; z = 5 + 8k \; ✓$$

Since R is on the plane ✓

$$4x + 5y + 8z = 80$$

$$\therefore \quad 4(10 + 4k) + 5(8 + 5k) + 8(5 + 8k) = 80 ✓$$

$$\therefore \quad 40 + 16k + 40 + 25k + 40 + 64k = 80$$

$$105k = -40$$

$$\therefore \quad k = -\frac{40}{105}$$

$$\text{i.e. } k = -\frac{8}{21} ✓$$

[4]

g. Hence, or otherwise, determine the position vector for the point R such that R is the closest point in the plane to the point P.

$$\overrightarrow{OR} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 + 4k \\ 8 + 5k \\ 5 + 8k \end{pmatrix} ✓$$

$$= \begin{pmatrix} \frac{178}{21} \\ \frac{128}{21} \\ \frac{41}{21} \end{pmatrix} ✓ \qquad \begin{matrix} 8.476\ldots \\ 6.095\ldots \\ 1.952\ldots \end{matrix}$$

[2]

h. Give the shortest distance, correct to 0.01 cm, from point P to the plane.

$$\overrightarrow{PR} = \begin{pmatrix} -\frac{32}{21} \\ -\frac{40}{21} \\ -\frac{64}{21} \end{pmatrix} \qquad \therefore \quad |\overrightarrow{PR}| = \frac{8\sqrt{105}}{21} = 3.90 \text{ cm} ✓ \; (2\text{ dp})$$

Σ 11m

[1]

2. A plane is flying in a straight line with velocity vector $k\mathbf{i} + 50\mathbf{j} - 2\mathbf{k}$ km/hr. It is noticed that a car is travelling (on the ground) with velocity $100\mathbf{j}$ km/hr.

At 1030 hrs the car is at position $100\mathbf{i} - 80\mathbf{j}$ km, whilst the plane is at $40\mathbf{j} + 5\mathbf{k}$ km.

a. Determine the expression for the position of the plane from the car at time t hours after 1030 hrs.

$$\underset{\sim}{r}_p = \begin{pmatrix} 0 \\ 40 \\ 5 \end{pmatrix} + t\begin{pmatrix} k \\ 50 \\ -2 \end{pmatrix}, \quad \underset{\sim}{r}_c = \begin{pmatrix} 100 \\ -80 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 100 \\ 0 \end{pmatrix} \checkmark$$

$$\therefore\ {}_p\underset{\sim}{r}_c = \underset{\sim}{r}_p - \underset{\sim}{r}_c \checkmark$$

$$= \begin{pmatrix} -100 \\ 120 \\ 5 \end{pmatrix} + t\begin{pmatrix} k \\ -50 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -100 + kt \\ 120 - 50t \\ 5 - 2t \end{pmatrix} \checkmark$$

[3]

A neutral stationary observer notices that the plane is travelling approximately in a north-easterly direction.

b. If it is known that the plane is closest to the car at exactly 1200 hrs, determine the value(s) of the constant k correct to 0.001.

Closest approach occurs at $t = 1{\cdot}5$ hrs $\checkmark$ when

$${}_p\underset{\sim}{r}_c \bullet {}_p\underset{\sim}{V}_c = 0 \checkmark$$

$$\therefore \begin{pmatrix} -100 + kt \\ 120 - 50t \\ 5 - 2t \end{pmatrix} \bullet \begin{pmatrix} k \\ -50 \\ -2 \end{pmatrix} = 0$$

$\therefore$ For $t = 1{\cdot}5$ hrs

$$\begin{pmatrix} -100 + 1{\cdot}5k \\ 45 \\ 2 \end{pmatrix} \checkmark \bullet \begin{pmatrix} k \\ -50 \\ -2 \end{pmatrix} = 0$$

$$\therefore\ k(1{\cdot}5k - 100) - 45(50) - 4 = 0$$

$$1{\cdot}5k^2 - 100k - 2\,254 = 0 \checkmark$$

Solving gives $k = -17{\cdot}791\ldots$

or $k = 84{\cdot}458\ldots$

Σ 9m

There is more space for part b. over page

[6]

2. b. If it is known that the plane is closest to the car at exactly 1200 hrs, determine the value(s) of the constant k correct to 0.001.

As plane moves approx. in a NE direction, then $k > 0$.

$\therefore$ Reject $k = -17.791\ldots$ ✓

$\therefore$ $k = 84.458\ldots$ ✓ for closest approach at 12 noon.

[6]

c. Give the distance of the closest approach correct to the nearest metre.

Closest distance $= \left| {}_{P}\underset{\sim}{r}_{C} \right|$ ✓ at $t = 1.5$ hrs, $k = 84.458\ldots$

$$= \left| \begin{pmatrix} 26.687\ldots \\ 45 \\ 2 \end{pmatrix} \right|$$

$$= 52.3567\ldots \text{ km}$$

$\therefore$ Closest approach is 52.357 km ✓ (3 d.p.)

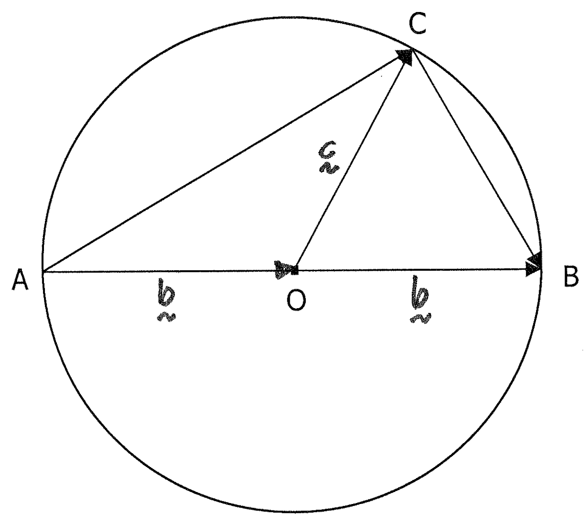
[2]

Σ 2m

In questions 3 and 4, each must be proved using vector methods.

3. A circle is given with centre O. Points A and B form a diameter of the circle, with point C being any point of the semi-circle.

Prove that s∠ ACB = 90°.



Prove : $\overrightarrow{AC} \perp \overrightarrow{CB}$

Hint : Let $\overrightarrow{OB}$ = **b** and $\overrightarrow{OC}$ = **c**

$\overrightarrow{AC} = \underset{\sim}{b} + \underset{\sim}{c}$ ✓

$\overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c}$

∴ Consider $\overrightarrow{AC} \cdot \overrightarrow{CB}$ ✓

$= (\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} - \underset{\sim}{c})$

$= \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{c}$ ✓

$= |\underset{\sim}{b}|^2 - |\underset{\sim}{c}|^2$ ✓

But since $|\underset{\sim}{b}| = |\underset{\sim}{c}|$ ✓ congruent radii

∴ $\overrightarrow{AC} \cdot \overrightarrow{CB} = |\underset{\sim}{b}|^2 - |\underset{\sim}{b}|^2$

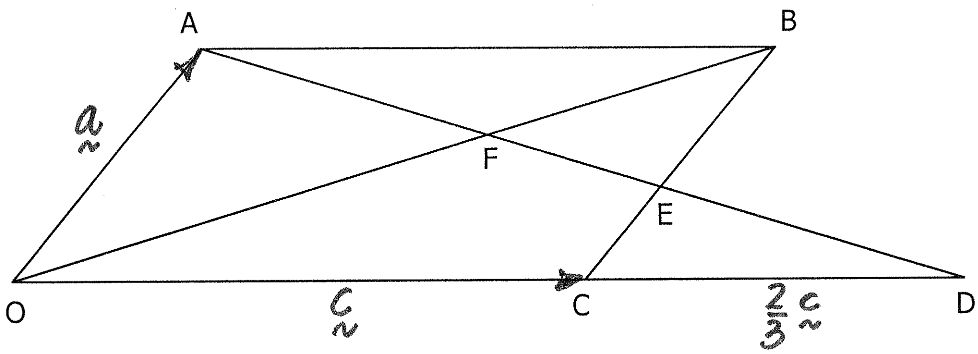
$= 0$ ✓

∴ $\overrightarrow{AC} \perp \overrightarrow{CB}$

∴ s∠ ACB = 90°.

[6]

4. Parallelogram OABC is shown, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$. $\overrightarrow{OC}$ is extended to determine point D such that $3\overrightarrow{CD} = 2\overrightarrow{OC}$. $\overrightarrow{AD}$ intersects $\overrightarrow{BC}$ at point E whilst $\overrightarrow{OB}$ intersects $\overrightarrow{AD}$ at point F.



a. Express $\overrightarrow{OD}$ in terms of $\overrightarrow{OC}$.

$\overrightarrow{OD} = \underset{\sim}{c} + \frac{2}{3}\underset{\sim}{c} = \frac{5}{3}\underset{\sim}{c}$ ✓

[1]

Let $\overrightarrow{AE} = k\,\overrightarrow{AD}$ and $\overrightarrow{BE} = m\,\overrightarrow{BC}$, where k, m are real constants.

b. Write an expression for $\overrightarrow{AD}$ in terms of vectors **a** and **c**

$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$

$= -\underset{\sim}{a} + \frac{5}{3}\underset{\sim}{c}$ ✓ ✓

[2]

c. Given that $\overrightarrow{AE} = k\,\overrightarrow{AD}$ and $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$, obtain TWO expressions for the vector $\overrightarrow{AE}$, and hence prove that $k = m = 0.6$

$\overrightarrow{AE} = k\left(-\underset{\sim}{a} + \frac{5}{3}\underset{\sim}{c}\right)$ ✓

Also $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$

$= \underset{\sim}{c} + m(-\underset{\sim}{a})$

$= -m\underset{\sim}{a} + \underset{\sim}{c}$ ✓

✓ Equating expressions for $\overrightarrow{AE}$:

$-k\underset{\sim}{a} + \frac{5k}{3}\underset{\sim}{c} = -m\underset{\sim}{a} + \underset{\sim}{c}$

$\therefore$ $k = m$ and $\frac{5k}{3} = 1$ ✓

$\therefore$ $k = \frac{3}{5}$ ✓

$\therefore$ $m = \frac{3}{5} = 0.6$

[5]

Σ 8m

4. d. Prove that $\overrightarrow{AF} = \frac{3}{8}\overrightarrow{AD}$ and $\overrightarrow{OF} = \frac{5}{8}\overrightarrow{OB}$.

Let $\overrightarrow{AF} = x\overrightarrow{AD}$, $\overrightarrow{OF} = y\overrightarrow{OB}$

As $\overrightarrow{AD} = -\underset{\sim}{a} + \frac{5}{3}\underset{\sim}{c}$

$\therefore \overrightarrow{AF} = -x\underset{\sim}{a} + \frac{5x}{3}\underset{\sim}{c}$ ✓

Also $\overrightarrow{AF} = \overrightarrow{AO} + \overrightarrow{OF}$

$= -\underset{\sim}{a} + y(\underset{\sim}{a} + \underset{\sim}{c})$ ✓

$= (y-1)\underset{\sim}{a} + y\underset{\sim}{c}$ ✓

✓ Equating expressions for $\overrightarrow{AF}$:

$-x\underset{\sim}{a} + \frac{5x}{3}\underset{\sim}{c} = (y-1)\underset{\sim}{a} + y\underset{\sim}{c}$

$\therefore -x = y-1$ and $\frac{5x}{3} = y$ ✓

Solving gives $x = \frac{3}{8}$

$y = \frac{5}{8}$ ✓

$\therefore \overrightarrow{AF} = \frac{3}{8}\overrightarrow{AD}$

$\overrightarrow{OF} = \frac{5}{8}\overrightarrow{OB}$.

[6]

Σ 6m

End of Assessment Task